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Slab Percolation and Phase Transitions for the Ising Model

Emilio De Santis[1] and Rossella Micieli[1]

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We prove, using the random-cluster model, a strict inequality between site percolation and magnetization in the region of phase transition for the d-dimensional Ising model, thus improving a result of [5]. We extend this result also at the case of two plane lattices $\mathbb{Z}^2$ (slabs) and give a characterization of phase transition in this case. The general case of N slabs, with N an arbitrary positive integer, is partially solved and it is used to show that this characterization holds in the case of three slabs with periodic boundary conditions.

KEY WORDS: Percolation; infinite clusters; magnetization; Gibbs measure; random-cluster measure.
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1. INTRODUCTION

At the end of the 70's the seminal paper [5] showed the connection between phase transition for the ferromagnetic Ising model and site percolation. This point of view has given a geometrical interpretation of phase transition, initiating a new line of research. Following this approach Higuchi developed techniques to study percolation for the two dimensional Ising model, with non zero external field [18, 19, 20, 21]. For example, he showed that for every $\beta < \beta_c$ there exists a positive critical point $h_c(\beta)$ such that an infinite cluster of $(+)$-sites does not exist for all $h < h_c$. In [2] the authors study percolation on $\mathbb{Z}^d$; they show that for $h > h_c(0) + 2d\beta$, or for $h \geq 0$ when $\beta > \beta_c$ there is percolation.

There are substantial differences between dimensions two and three for percolation in the Ising model; in fact in [7] it is showed that in three dimensions there is coexistence of infinite plus and minus clusters (at least for small values of

[1] Dipartimento di Matematica "Guido Castelnuovo" Università di Roma "La Sapienza". P.le Aldo Moro 2,00185 Roma, Italia; e-mail: desantis@mat.uniroma1.it

the parameter β), while in two dimensions it is proved that the infinite clusters of opposite sign can not coexist [5, 14].

In our paper there are two main results. The first is a strict inequality between percolation probability and magnetization, and the second one is an extension of [5].

The results have a common spirit, in fact, using also [2], we can show that for dimensions large enough there is percolation for each value of the temperature parameter. In contrast we show that, for some slab lattices, there is *column percolation* only in the region of transition phase.

To explain the physical meaning of the strict inequality let ρ^+_∞, ρ^+_f and ρ^- denote respectively the percentage of sites belonging to the positive infinite cluster, the percentage of sites belonging to positive finite clusters and the percentage of negative sites. These quantities, by spatial ergodicity, are well defined and $\rho^+_\infty + \rho^+_f + \rho^- = 1$. Then our inequality can be rewritten as $\rho^- > \rho^+_f$ when the Ising model is in the transition phase region with positive boundary conditions.

In the second part we deal with Ising model on slab graphs. The experimental and theoretical study of materials in thin-film form is an important subject in condensed matter field. Our paper could be a first step towards the study of thin crystals from a mathematical point of view. We characterize, using percolation, the transition phase of the Ising model on some slab lattices (thin crystal).

Now we show two examples in which the change of lattice from $\mathbb{Z}^2$ to $\mathbb{Z}^2 \times \{0, 1, \ldots, N\}$ has a different behavior.

The symmetric neighbor random walk on $\mathbb{Z}^2 \times \{0, 1, \ldots, N\}$ is recurrent for each value of N so it will have a different behavior with respect to the symmetric neighbor random walk on $\mathbb{Z}^3$. Vice versa the non coesistence result, under the general hypotheses in [14], of minus and plus infinite clusters on $\mathbb{Z}^2$ is lost on the slab lattices $\mathbb{Z}^2 \times \{0, 1, \ldots, N\}$.

From the fact that in physics there are few really 2-dimensional crystal (lattice) we are interested to study properties that are stable in slab lattices (at least for small N) and in our paper we see that for small value of N percolation characterize the phase transition also in these interesting lattices.

In the last twenty years the physicists have done, studing thin matter, a large number of observations. We believe that an effort should be done to understand some new phenomena also connect with Statistical Mechanics. For example an interesting question, in part related with our result, could be the study of the Dobrushin b.c. on slabs lattices or on lattices where the thermodynamic limit is taken with a different speed on the axes $\hat{x}$, $\hat{y}$, and $\hat{z}$. In the next years an effort should be done to study, from a mathematical point of view, the physics of thin-crystal.

The paper is organized as follows: in Section 2 we set the notation and introduce some basic tools. In Section 3 we use the random cluster model to give, below the critical temperature, a strict inequality between magnetization and site percolation probability in the d-dimensional cubic lattice, in this way we improve a result of [5] (similar argument are used in [12]). Then, in Section 4 we partially generalize the result to some *slab graphs*. Such graphs, also called *bunkbed graphs*, have attracted the attention of other researches (see [3, 16]) in the study of random walks, random-cluster model and some correlation inequalities for the Ising model. In Section 5, for $N = 2$, we prove a characterization of phase transitions similar to that for the $\mathbb{Z}^2$ lattice, thereby obtaining an inequality between magnetization and percolation probability of *columns* formed only of $+1$ spins. For $N = 3$ and periodic boundary conditions, we are able to characterize the phase transition through percolation of columns with majority of plus spins. However we cannot obtain any meaningful inequality in this case.

2. BASIC DEFINITIONS AND NOTATION

In this section we set our notation for percolation, ferromagnetic Ising model, and random-cluster model.

Let $d \geq 2$ and let $\mathbb{Z}^d$ be the set of all points $x = (x_1, x_2, \ldots, x_d)$ with integral coordinates. The distance $||x - y||$ between x and y is defined by $||x - y|| = \sum_{i=1}^{d} |x_i - y_i|$. If $||x - y|| = 1$ we say that x and y are *adjacent*. We turn $\mathbb{Z}^d$ into a graph, called the *d-dimensional cubic lattice*, by adding edges $e = \langle x, y\rangle$ between all pairs x, y of adjacent points of $\mathbb{Z}^d$; we denote this lattice by $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, where $\mathbb{E}^d$ is the edge set. The edge $e = \langle x, y\rangle$ is said to be *incident* to the vertices x and y in this case we also say that x and y are endvertices of the edge $e \in \mathbb{E}^d$.

Let Λ be a box of $\mathbb{Z}^d$, *i.e.* $\Lambda = \prod_{i=1}^{d}[x_i, y_i]$ for some $x, y \in \mathbb{Z}^d$, where $[x_i, y_i]$ is the set $\{x_i, x_i + 1, x_i + 2, \ldots, y_i\}$; we write $\mathbb{E}_\Lambda$ for the set of edges $e = \langle x, y\rangle$ in $\mathbb{E}^d$ such that $x, y \in \Lambda$.

A *path* of $\mathbb{L}^d$ is an alternating sequence $(x_0, e_0, \ldots, e_{n-1}, x_n)$ of distinct vertices and edges with $e_i = \langle x_i, x_{i+1}\rangle$ for all $i = 0, \ldots, n-1$; such a path has *length n* and is said to *connect* x_0 to x_n. A subset $Y \subset \mathbb{Z}^d$ is *connected* if for all pairs x, y of vertices in Y, there exists a path connecting the vertices x, y having all its vertices belonging to Y. The *boundary* of $Y \subset \mathbb{Z}^d$ is the set ∂Y of all vertices in $\mathbb{Z}^d \setminus Y$ that are adjacent to at least one vertex in Y.

The *edge space* is $\Omega = \{0, 1\}^{\mathbb{E}^d}$, elements of which are $\omega = (\omega(e) : e \in \mathbb{E}^d)$. We say that the edge e is *open* if $\omega(e) = 1$, and *closed* if $\omega(e) = 0$. An *open cluster* of ω is a maximal connected component of $\omega^{-1}(1)$. We write $\{x \leftrightarrow \infty\}$ for the set of configurations $\omega \in \Omega$ such that x belongs to an infinite open cluster of ω.

Let $\Sigma = \{-1, +1\}^{\mathbb{Z}^d}$ be the *spin space*, elements of which are $\sigma = (\sigma_x : x \in \mathbb{Z}^d)$; σ_x is the spin on x. For $\sigma \in \Sigma$ a $(+)$-*cluster* of σ is a maximal connected component of $\sigma^{-1}(1)$. A $(-)$-*cluster* is defined in a similar way. We use the notation $(\infty, \pm)$-cluster to indicate an infinite $(\pm)$-cluster.

For $0 \leq p \leq 1$, let ϕ_p^1 be the random-cluster measure on Ω with wired boundary conditions (see [15]), and let $\mu_{\pm,\beta,J}$ (or simply $\mu_\pm$) be the Ising measure on Σ with $(\pm)$-boundary conditions, zero external field ($h = 0$) and interactions $\{J_e\}_{e\in\mathbb{E}^d}$

(see [22]). In this paper we agree that on each edge e of the graph under consideration there is a constant interaction $J_e \equiv J = 1$ (ferromagnetic Ising model). In the paper we will write *Ising model* in place of *ferromagnetic Ising model with zero external field*. In some cases one could take different values of the interactions on different edges; this will be partially discussed in the last section.

For $p = 1 - \exp\{-2\beta\}$ let us denote with $\nu_{\pm,\beta}$ (or simply $\nu_{\pm}$) the *coupling between Ising and random-cluster measures* on $\Sigma \times \Omega$ with plus boundary conditions as defined in [23] (see also [8]).

For details in the construction and relations between the three measures ν_+, μ_+ and ϕ_p^1 see [23]. Newman also deal with a more general setting of boundary conditions and no-ferromagnetic interactions.

We give some other definitions. We put $C_\infty^\pm = \{\sigma \in \Sigma : 0 \in (\infty, \pm)\text{-cluster of } \sigma\}$, where 0 denotes the origin of $\mathbb{Z}^d$. The *percolation probability* is denoted by $R(\pm; \mu_\pm) = \mu_\pm(C_\infty^\pm)$, and the *magnetization in the origin* is

$$M(\mu_\pm) = \mathbb{E}_{\mu_\pm}(\sigma_0) = \mu_\pm(\sigma_0 = +1) - \mu_\pm(\sigma_0 = -1). \tag{1}$$

It is known that

$$M(\mu_+) = \phi_p^1(0 \leftrightarrow \infty), \tag{2}$$

(see [15, 23]).

3. $\mathbb{Z}^d$ PERCOLATION AND MAGNETIZATION

In this section we prove that if the temperature is lower than the critical temperature, or equivalently if the magnetization is positive, then the percolation probability is strictly greater than the magnetization and we give an explicit estimate.

Theorem 3.1. *For the Ising model, the following relation holds:*

$$R(\pm; \mu_\pm) \geq |M(\mu_\pm)| + \frac{1}{2}|M(\mu_\pm)| \left(\frac{p}{2-p}\right)^{2d(3^d-1)} (1-p)^{2d}, \tag{3}$$

where $p = 1 - \exp\{-2\beta\}$.

Proof: Let

$$\Lambda' = \{x \in \mathbb{Z}^d \setminus \{0\} : -1 \leq x_i \leq 1 \text{ for all } i = 1, \dots, d\}.$$

We consider the following cylinders on Λ'

$$A = \{\omega \in \Omega : \omega(e) = 1 \text{ for } e \in \mathbb{E}_{\Lambda'},\ \omega(e) = 0 \text{ for } e = \langle 0, y\rangle, y \in \Lambda'\},$$

$$B = \{\omega \in \Omega : \omega(e) = 1 \text{ for } e \in \mathbb{E}_{\Lambda'},\ \omega(e) = 1 \text{ for } e = \langle 0, y\rangle, y \in \Lambda'\}.$$

Note that the event A (resp. B) forces all edges in $\mathbb{E}_{\Lambda'}$ to be open, and all edges incidents at the origin to be closed (resp. open).

Let x_0 be a vertex adjacent to the origin then, noting that the events $\{\sigma_0 = +1\} \times (\{x_0 \leftrightarrow \infty\} \cap A)$ and $\Sigma \times \{0 \leftrightarrow \infty\}$ are disjoint, we obtain

$$\nu_+(C_\infty^+ \times \Omega) \geq \nu_+(\Sigma \times \{0 \leftrightarrow \infty\}) + \nu_+(\{\sigma_0 = +1\} \times \{\{x_0 \leftrightarrow \infty\} \cap A\}),$$

thus

$$\nu_+(C_\infty^+ \times \Omega) \geq \nu_+(\Sigma \times \{0 \leftrightarrow \infty\}) + \frac{1}{2}\nu_+(\Sigma \times (\{x_0 \leftrightarrow \infty\} \cap A)).$$

Hence,

$$\begin{aligned} R(+;\mu_+) = \mu_+(C_\infty^+) &\geq \phi_p^1(0 \leftrightarrow \infty) + \frac{1}{2}\phi_p^1(\{x_0 \leftrightarrow \infty\} \cap A) \\ &= M(\mu_+) + \frac{1}{2}\phi_p^1(\{x_0 \leftrightarrow \infty\} \cap A). \end{aligned} \tag{4}$$

For the structure of the random cluster measure we obtain

$$\phi_p^1(x_0 \leftrightarrow \infty \mid A) = \phi_p^1(x_0 \leftrightarrow \infty \mid B). \tag{5}$$

Events $\{x_0 \leftrightarrow \infty\}$ and B are increasing, thus by FKG inequality [9] we obtain

$$\phi_p^1(x_0 \leftrightarrow \infty \mid B) \geq \phi_p^1(x_0 \leftrightarrow \infty). \tag{6}$$

By (5) and (6) follows

$$\phi_p^1(\{x_0 \leftrightarrow \infty\} \cap A) \geq \phi_p^1(x_0 \leftrightarrow \infty)\,\phi_p^1(A) = M(\mu_+)\phi_p^1(A), \tag{7}$$

where the last equality follows by the translation invariance of ϕ_p^1.

We give now a lower bound for $\phi_p^1(A)$. Let $k(d, \Lambda')$ be the number of edges of the graph $(\Lambda', \mathbb{E}_{\Lambda'})$. For $\omega_{\setminus e} \in \Omega_{\setminus e} = \{0, 1\}^{\mathbb{E}^d \setminus \{e\}}$, we have [10] (see also [15, 23])

$$\phi_p^1(\omega(e) = 1 \mid \omega_{\setminus e}) \in \left\{p, \frac{p}{2-p}\right\}. \tag{8}$$

Moreover

$$\phi_p^1(A) = \phi_p^1(\{\omega(e) = 1, e \in \mathbb{E}_{\Lambda'}\} \cap \{\omega(e) = 0, e = \langle 0, y\rangle, y \in \Lambda'\}), \tag{9}$$

thus, by (8) and (9) follows

$$\phi_p^1(A) \geq \left(\frac{p}{2-p}\right)^{k(d,\Lambda')} (1-p)^{2d}. \tag{10}$$

By (7) and (10), we obtain

$$\phi_p^1(\{x_0 \leftrightarrow \infty\} \cap A) \geq M(\mu_+)\left(\frac{p}{2-p}\right)^{k(d,\Lambda')} (1-p)^{2d}. \tag{11}$$

We also give an upper bound for $k(d, \Lambda')$. The number of vertices in Λ' is $3^d - 1$ and there are at most $2d$ edges incident to each vertex in Λ', so $k(d, \Lambda') \leq 2d(3^d - 1)$. Thus, by (4) and (11), Theorem 3.1 is proved. □

We notice that the bound of Theorem 3.1 deteriorates in high dimensions. It is actually known that the percolation is enhanced for large dimensions; for example in [2] is proved that for large d the probability of percolation remains positive for all values of $\beta > \beta_d$. Using [2] one could improve the inequality (3) showing that for dimension d large enough there is percolation for each value of the temperature parameter. We end the section with this remark.

Remark 31. The Onsager solution for the two dimensional ferromagnetic Ising model shows the exact value of magnetization as a function of $\beta \in [\beta_c, \infty)$ [24]. It is

$$M(\mu_+) = \{1 - [\sinh(2\beta)]^{-4}\}^{\frac{1}{8}}. \tag{12}$$

We can re-write (12) as a function of the parameter $x = 1 - p = \exp\{-2\beta\}$ obtaining

$$M(\mu_+) = \left\{1 - \left[\frac{2x}{1 - x^2}\right]^4\right\}^{\frac{1}{8}}. \tag{13}$$

Then, using Taylor expansion we obtain $m = 1 - 2x^4 + o(x^4)$, giving the magnetization for small values of the temperature (corresponding to small x). We do not have an explicit formula for the percolation probability but for small x it is easy to calculate the first terms in Taylor expansion. We find

$$R(+; \mu_+) = 1 - x^4 + o(x^4). \tag{14}$$

This general relation also holds for regular graphs

$$(1 - R(+; \mu_+)) \sim 1/2(1 - M(\mu_+)) \sim x^n$$

where n is the degree of the origin.

4. N SLABS PERCOLATION AND MAGNETIZATION

In this section we propose a conjecture for the characterization of phase transition through percolation in the case of N slabs and we prove some general results.

We introduce some basic definitions for slabs. Let $\mathbb{Z}^2$ be the two-dimensional lattice, and consider the set $\mathbb{Z}^2 \times \{0, 1, \ldots, N - 1\}$, where N is an arbitrary positive integer. We put an edge between each pair of vertices having unitary distance. Its edge set is denoted by $\mathbb{E}^{2,N}$.

Definition 42. An N*-vertex* $\mathbf{c}_{i,j}$ is a vector

$$\mathbf{c}_{i,j} = ((i, j, 0), (i, j, 1), \ldots, (i, j, N-1))$$

where $i, j \in \mathbb{Z}$. An N*-edge* $\mathbf{e}$ is formed by a couple of N-vertices $< \mathbf{c}_{i,j}, \mathbf{c}_{l,m} >$ where the vertices $(i, j), \ (l, m) \in \mathbb{Z}^2$ are adjacent.

We give analogous definitions to section 2 just substituting vertices by N-vertices so we write N*-set, Npath*, etc. We denote with Γ the family of all finite N-connected N-set containing the N-vertex at the origin $\mathbf{c}_{0,0}$.

We define $\Sigma^{(N)} = \{-1, +1\}^{\mathbb{Z}^2 \times \{0,1,\ldots,N-1\}}$, and for $\sigma \in \Sigma^{(N)}$ we indicate with $\sigma_{i,j,k} \in \{-1, +1\}$ the spin on the vertex (i, j, k).

Definition 43. For $\sigma \in \Sigma^{(N)}$, a $(\mathbf{c}^+)$*-cluster* ($(\mathbf{c}^-)$*-cluster*) of σ is a maximal N-connected component of N-vertices $\mathbf{c_{i,j}}$ such as $\sum_{k=0}^{N-1} \sigma_{i,j,k} > 0 (\sum_{k=0}^{N-1} \sigma_{i,j,k} < 0)$. We write $(\infty, \mathbf{c}^{\pm})$-cluster for an infinite $(\mathbf{c}^{\pm})$-cluster.

We set the event

$$N\text{-}C_\infty^{\pm} = \{\sigma \in \Sigma^{(N)} : \mathbf{c}_{0,0} \in \ (\infty, \mathbf{c}^{\pm})\text{- cluster of } \sigma\},$$

Notice that N-C_∞^+ is the event that the N-origin belongs to an infinite $(\mathbf{c}^+)$-cluster. Let E^+ (E^-) be the event that the N-origin has a majority of spins +1 (-1) on its vertices; obviously the events E^+ and E^- are disjoint and, for odd values of N, $E^+ \cup E^- = \Sigma^{(N)}$.

Let $\mu_{\pm}$ be the Ising measure on $\Sigma^{(N)}$ with $(\pm)$-boundary conditions. We set up the interactions equal to one between all spins on two adjacent vertices. The N*-percolation probability* is $R(\mathbf{c}^{\pm}; \ \mu_{\pm}) = \mu_{\pm}(N\text{-}C_\infty^{\pm})$.

In the next proposition we show that if the N-percolation probability is positive then magnetization is positive.

Proposition 4.3. *For the Ising model on $\Sigma^{(N)}$, the following relation holds:*

$$R(\mathbf{c}^{\pm}; \mu_{\pm}) > 0 \Rightarrow |M(\mu_{\pm})| > 0.$$

Proof: We project the N slabs on a single lattice $\mathbb{Z}^2$ by assigning spins +1 (-1) on the vertices corresponding to N-vertices with a majority of spins +1 (-1) and choosing spins +1 or -1 with probability $\frac{1}{2}$ on the remaining vertices. This construction induces a new measure $\pi_{\pm}$ on $\Sigma = \{-1, +1\}^{\mathbb{Z}^2}$. We note that if there exists an infinite $(\mathbf{c}^+)$-cluster in $\mathbb{Z}^2 \times \{0, 1, \ldots, N-1\}$, then there exists an infinite $(+)$-cluster in the new lattice. If $M(\mu_{\pm}) = 0$ then $M(\pi_{\pm}) = 0$. By using the result of non coexistence in two dimensions of an $(\infty, +)$-cluster with an $(\infty, -)$-cluster given in [14] (see also [12]) follows $R(\pm; \ \pi_{\pm}) = 0$. Thus also $R(\mathbf{c}^{\pm}; \ \mu_{\pm}) = 0$ by the observation above. □

The opposite implication of Proposition 4.3 will be partially proved.

Lemma 4.4. *Let $(H, \mathcal{A}, \mathbb{P})$ be an arbitrary probability space. If X and Y are random variables with X symmetric and Y not negative, then*

$$\mathbb{P}(X + Y > 0) \geq \mathbb{P}(X + Y < 0).$$

Proof: Since $Y \geq 0$, $\{X > 0\} \subseteq \{X + Y > 0\}$ and $\{X + Y < 0\} \subseteq \{X < 0\}$. Thus, because of X is symmetric

$$\mathbb{P}(X + Y > 0) \geq \mathbb{P}(X > 0) = \mathbb{P}(X < 0) \geq \mathbb{P}(X + Y < 0).$$

□

The following proposition says that if there is phase transition then, with $(+)$-boundary conditions, the probability of a majority of $+1$ spins on the N-origin is larger than the probability of a majority of -1 spins.

Proposition 4.5. *For the Ising model on $\Sigma^{(N)}$ we have: $|\mu_{\pm}(E^+) - \mu_{\pm}(E^-)| > 0$ if and only if $\beta > \beta_c$.*

Proof: Suppose $\beta \leq \beta_c$, then $\mu_+ = \mu_- = \mu$, so

$$\mu_{\pm}(E^+) = \mu(E^+) = \mu(E^-) = \mu_{\pm}(E^-).$$

Conversely, let us consider $\omega \in \Omega^{(N)} = \{0, 1\}^{\mathbb{E}^{2,N}}$. Given $\omega \in \Omega^{(N)}$, the sum of spins on the vertices in $\mathbf{c}_{0,0}$ can be expressed as the sum of a symmetric random variable (vertices belonging to a finite cluster of open edges) and a positive random variable (vertices belonging to an infinite cluster of open edges), thus by Lemma 4.4

$$\nu_+(E^+ \times \Omega^{(N)} \,|\, \omega) \geq \nu_+(E^- \times \Omega^{(N)} \,|\, \omega). \tag{15}$$

We have

$$\mu_+(E^+) = \int_{\Omega^{(N)}} \nu_+(E^+ \times \Omega^{(N)} \,|\, \omega)\, \phi_p^1(d\omega). \tag{16}$$

Analogously for $\mu_+(E^-)$. Let A be the event that all vertices in $\mathbf{c}_{0,0}$ belong to an infinite open cluster. More precisely

$$A = \{\omega \in \Omega^{(N)} : (0, 0, k) \leftrightarrow \infty \text{ for all } k = 0, \ldots, N - 1\}.$$

Given $\omega \in A$, the conditional measure is obtained by setting $\sigma_{0,0,k} = +1$ for every $k = 0, \dots, N-1$, thus $\nu_+(E^+ \times \Omega^{(N)} \mid \omega) = 1$. Hence, by (15) and (16), follows

$$\begin{aligned}\mu_+(E^+) &= \int_A \nu_+\big(E^+ \times \Omega^{(N)} \,\big|\, \omega\big)\, \phi_p^1(d\omega) + \int_{\Omega^{(N)}\setminus A} \nu_+\big(E^+ \times \Omega^{(N)} \mid \omega\big)\, \phi_p^1(d\omega) \\ &= \int_A \phi_p^1(d\omega) + \int_{\Omega^{(N)}\setminus A} \nu_+\big(E^+ \times \Omega^{(N)} \,\big|\, \omega\big)\, \phi_p^1(d\omega) \\ &\geq \phi_p^1(A) + \int_{\Omega^{(N)}\setminus A} \nu_+\big(E^- \times \Omega^{(N)} \,\big|\, \omega\big)\, \phi_p^1(d\omega). \end{aligned} \tag{17}$$

Moreover if $\omega \in A$ then $\nu_+(E^- \times \Omega^{(N)} \mid \omega) = 0$. Thus

$$\mu_+(E^-) = \int_{\Omega^{(N)}\setminus A} \nu_+(E^- \times \Omega^{(N)} \mid \omega)\, \phi_p^1(d\omega). \tag{18}$$

By using (15), (17) and (18), we obtain

$$\mu_+(E^+) - \mu_+(E^-) \geq \phi_p^1(A). \tag{19}$$

Consider now the events

$$F = \{\omega \in \Omega^{(N)} : (0,0,0) \leftrightarrow \infty\},$$

$$G = \{\omega \in \Omega^{(N)} : \omega(e) = 1 \text{ for } e = \langle (0,0,k-1), (0,0,k)\rangle,\ k = 1, .., N-1\}.$$

We note that $A \supseteq F \cap G$. Since F and G are increasing events, by FKG inequality we obtain

$$\phi_p^1(A) \geq \phi_p^1(F \cap G) \geq \phi_p^1(F)\, \phi_p^1(G). \tag{20}$$

But, by hypothesis, $\phi_p^1(F) = \phi_p^1((0,0,0) \leftrightarrow \infty) = M(\mu_+) > 0$ and $\phi_p^1(G) > 0$ depending on a finite number of edges. By inequality (20) we get $\phi_p^1(A) > 0$, hence

$$\mu_+(E^+) - \mu_+(E^-) \geq \phi_p^1(A) > 0.$$

The same argument holds for $(-)$-boundary conditions, therefore

$$\text{Phase transition} \Leftrightarrow |M(\mu_\pm)| > 0 \Rightarrow |\mu_\pm(E^+) - \mu_\pm(E^-)| > 0.$$

$\square$

We are now in the position to present our conjecture for the characterization of phase transition for the Ising model on the lattices $\mathbb{Z}^2 \times \{0, 1, \dots, N-1\}$. We believe that, for these models, N-percolation probability is positive if and only if there is phase transition. Proposition 4.3 shows that an implication is true. To prove the other one we should use Proposition 4.5 and the next argument.

Let $Y \in \Gamma$ and set

$$C_Y^{\pm} = \{\sigma \in \Sigma^{(N)} : Y \in \Gamma \text{ is a } (\mathbf{c}^{\pm})\text{-cluster of } \sigma\}. \tag{21}$$

We have, as in [5]

$$\mu_+(E^+) - \mu_+(E^-) = \mu_+(E^+) - \mu_-(E^+) = \tag{22}$$

$$= \sum_{Y \in \Gamma} (\mu_+(C_Y^+) - \mu_-(C_Y^+)) + \mu_+(N\text{-}C_\infty^+) - \mu_-(N\text{-}C_\infty^+).$$

Thus, a sufficient condition for the claim is

$$\mu_+(C_Y^+) \le \mu_-(C_Y^+), \qquad \text{for all } Y \in \Gamma. \tag{23}$$

Indeed, by assumption (23) and Proposition 4.5 for $\beta > \beta_c$ one obtains

$$R(\mathbf{c}^+; \mu_+) = \mu_+(N\text{-}C_\infty^+) \ge \mu_+(E^+) - \mu_+(E^-) > 0. \tag{24}$$

Therefore Proposition 4.3 and inequality (24) give a characterization of phase transition through percolation in the case of N slabs. In the next section, we present the cases of two ($N = 2$) and three slabs ($N = 3$) with periodic boundary conditions, showing that (23) holds.

5. THE CASES OF TWO AND THREE SLABS

In this section, we give a characterization of phase transition through percolation in the cases of two and three slabs. We manage to the case of two slabs the result of [5]. The extension to the case of three slabs is done in a different flavor. We start with some new definitions.

Let B^+ be the event that each vertex in $\partial\Lambda$, where Λ is an N-box, has $+1$ spin, and similarly for B^-. If C_Y^+ is given by (21), let ∂C_Y^+ be the event such that each N-vertex in ∂Y does not have a majority of $+1$ spins on its vertices. In general we denote by

$$\sigma_V = \{\tilde{\sigma} \in \Sigma_\Lambda^{(N)} : \tilde{\sigma}_{i,j,k} = \sigma_{i,j,k} \text{ for all } \mathbf{c}_{i,j} \in V,\ k = 0, \dots, N-1\}$$

a cylinder where the values of $\sigma_{i,j,k} \in \{-1, +1\}$ are assigned on $V \subset \Lambda$. If V_1 and V_2 are two disjoint sets of vertices, we denote by $(\sigma_{V_1}, \sigma_{V_2})$ the cylinder $\sigma_{V_1 \cup V_2}$.

We remark that

$$\mu_\Lambda(B^+ \,|\, \sigma_X) \le \mu_\Lambda(B^+ \,|\, \bar{\sigma}_X), \quad \mu_\Lambda(B^- \,|\, \sigma_X) \ge \mu_\Lambda(B^- \,|\, \bar{\sigma}_X), \tag{25}$$

where $(\bar{\sigma}_X)_u \ge (\sigma_X)_u$ for every $u \in X$ (see [22]).

Theorem 5.1. *For the Ising model on $\Sigma^{(2)}$ the inequality $|M(\mu_\pm)| \le R(\mathbf{c}^\pm;\ \mu_\pm)$ holds. Moreover $R(\mathbf{c}^\pm;\ \mu_\pm) > 0$ if and only if $|M(\mu_\pm)| > 0$.*

Proof: For a given $Y \in \Gamma$, consider an N-box Λ_o such that $\Lambda_o \supset Y \cup \partial Y$. Let $\Lambda = \Lambda_o \cup \partial\Lambda_o$ and let μ_Λ be the Ising measure on $\Sigma_\Lambda^{(N)} = \{-1,\ +1\}^\Lambda$ with free boundary conditions.

Let us consider the cylinder $\mathbf{1}_Y \supset C_Y^+$ that assigns $+1$ spins to all the vertices in Y. For each $\sigma_{\partial Y} \subset \partial C_Y^+$, the cylinders $\{(\mathbf{1}_Y, \sigma_{\partial Y})\}_{\sigma_{\partial Y} \subset \partial C_Y^+}$ are all disjoint and $\bigcup_{\sigma_{\partial Y} \subset \partial C_Y^+} (\mathbf{1}_Y, \sigma_{\partial Y}) = C_Y^+$. We notice that C_Y^+ is not an increasing event because it gives information on each column belonging to ∂Y and precisely the sum of the spins in each column is less or equal to zero. Using also the Markov property for the Ising field we will obtain, in the end of the proof, that $\mu_{\Lambda_0}^+(C_Y^+) \le \mu_{\Lambda_0}^-(C_Y^+)$ for any finite set Y such that $Y \cup \partial Y \subset \Lambda_0$.

Consider the events B^+ and B^- referred to $\partial\Lambda_0$, then we can write

$$\begin{aligned}
\mu_{\Lambda_o}^+(C_Y^+) &= \frac{\mu_\Lambda(C_Y^+ \cap B^+)}{\mu_\Lambda(B^+)} = \frac{1}{\mu_\Lambda(B^+)} \sum_{\sigma_{\partial Y} \subset \partial C_Y^+} \mu_\Lambda((\mathbf{1}_Y, \sigma_{\partial Y}) \cap B^+) \\
&= \frac{1}{\mu_\Lambda(B^+)} \sum_{\sigma_{\partial Y} \subset \partial C_Y^+} \mu_\Lambda(B^+ \,|\, (\mathbf{1}_Y, \sigma_{\partial Y}))\mu_\Lambda((\mathbf{1}_Y, \sigma_{\partial Y})) \\
&= \frac{1}{\mu_\Lambda(B^+)} \sum_{\sigma_{\partial Y} \subset \partial C_Y^+} \mu_\Lambda(B^+ \,|\, \sigma_{\partial Y})\, \mu_\Lambda(\mathbf{1}_Y \,|\, \sigma_{\partial Y})\, \mu_\Lambda(\sigma_{\partial Y})
\end{aligned}$$

where we are using Markov property in the last equality.

Similarly for $\mu_{\Lambda_o}^-(C_Y^+)$. Since $\mu_\Lambda(B^+) = \mu_\Lambda(B^-)$, we have

$$\begin{aligned}
\frac{\mu_{\Lambda_o}^+(C_Y^+)}{\mu_{\Lambda_o}^-(C_Y^+)} &= \frac{\sum_{\sigma_{\partial Y} \subset \partial C_Y^+} \mu_\Lambda(B^+ \,|\, \sigma_{\partial Y})\, \mu_\Lambda(\mathbf{1}_Y \,|\, \sigma_{\partial Y})\, \mu_\Lambda(\sigma_{\partial Y})}{\sum_{\sigma_{\partial Y} \subset \partial C_Y^+} \mu_\Lambda(B^- \,|\, \sigma_{\partial Y})\, \mu_\Lambda(\mathbf{1}_Y \,|\, \sigma_{\partial Y})\, \mu_\Lambda(\sigma_{\partial Y})} \\
&\le \sup_{\sigma_{\partial Y} \subset \partial C_Y^+} \frac{\mu_\Lambda(B^+ \,|\, \sigma_{\partial Y})}{\mu_\Lambda(B^- \,|\, \sigma_{\partial Y})}. \qquad (26)
\end{aligned}$$

Let us define the event

$$F_{\partial Y} := \{\tilde{\sigma} \in \Sigma^{(2)} : (\tilde{\sigma}_{i,j,0}, \tilde{\sigma}_{i,j,1}) \in L\} \subset \partial C_Y^+, \qquad (27)$$

where $L = \{(-1, 1), (1, -1)\}$. By using (25) it is clear that the supremum in (26) is achieved for $\sigma_{\partial Y} \subset F_{\partial Y}$. Let us define the rotation operator $R : \Sigma \to \Sigma$ as:

$$(R\sigma)_{i,j,1} = \sigma_{i,j,0} \ \text{ and } \ (R\sigma)_{i,j,0} = \sigma_{i,j,1}.$$

The following equality is clear

$$\mu_\Lambda(B^+ | \sigma_{\partial Y}) = \mu_\Lambda(B^+ | (R\sigma)_{\partial Y}) \qquad (28)$$

because the first and second slab play the same role in the Ising model. Moreover if $\sigma_{\partial Y} \subset F_{\partial Y}$ then $(R\sigma)_{\partial Y} = -\sigma_{\partial Y}$, and in general is $\mu_\Lambda(B^+|\sigma_{\partial Y}) = \mu_\Lambda(B^-| - \sigma_{\partial Y})$.

Thus for each $\sigma_{\partial Y} \subset F_{\partial Y}$

$$\mu_\Lambda(B^+|\sigma_{\partial Y}) = \mu_\Lambda(B^+|(R\sigma)_{\partial Y}) = \mu_\Lambda(B^+| - \sigma_{\partial Y}) = \mu_\Lambda(B^-|\sigma_{\partial Y}). \tag{29}$$

Hence, by (29) and previous argument

$$\sup_{\sigma_{\partial Y} \subset \partial C_Y^+} \frac{\mu_\Lambda(B^+ \mid \sigma_{\partial Y})}{\mu_\Lambda(B^- \mid \sigma_{\partial Y})} = \sup_{\sigma_{\partial Y} \subset F_{\partial Y}} \frac{\mu_\Lambda(B^+ \mid \sigma_{\partial Y})}{\mu_\Lambda(B^- \mid \sigma_{\partial Y})} = 1. \tag{30}$$

Since this relation holds for all $\Lambda_o \supset Y \cup \partial Y$ then also in the limit $\Lambda_o \to \mathbb{Z}^2 \times \{0,\ 1\}$, by (26) and (30) we obtain (23).

To prove the inequality between 2-percolation probability and magnetization it is enough to observe that, by symmetry, we have $\mathbb{E}_{\mu_\pm}(\sigma_{0,0,0}) = \mathbb{E}_{\mu_\pm}(\sigma_{0,0,1})$, hence

$$M(\mu_\pm) = \frac{1}{2}\,\mathbb{E}_{\mu_\pm}(\sigma_{0,0,0} + \sigma_{0,0,1}) = \mu_\pm(E^+) - \mu_\pm(E^-).$$

Now, the first claim of the theorem immediately follows by (22) and (23). The second claim follows by the first inequality and Proposition 4.1. □

We present another case, in which we are able to prove (23), and thus to obtain characterization of phase transition via percolation. We consider the graph $\tilde{\mathcal{G}}_3$ having vertex set $\mathbb{Z}^2 \times \{0, 1, 2\}$ and edge set $\mathbb{E}^{2,3} \cup \mathbb{E}^p$, where $\mathbb{E}^p = \{\langle (i, j, 0), (i, j, 2)\rangle : i, j \in \mathbb{Z}\}$.

Consider the ferromagnetic Ising measures $\mu_\pm$ on $\tilde{\mathcal{G}}_3$, that we say to have *periodic boundary conditions*, and define

$$D_Y^+ = \{\sigma \in \Sigma^{(N)} : Y \in \Gamma \text{ belongs to a } (\mathbf{c}^+) - \text{cluster of } \sigma\}, \tag{31}$$

so that $C_Y^+ = D_Y^+ \cap \partial C_Y^+$. The event D_Y^+ depends only on the values of $\{\sigma_{i,j,k} : \mathbf{c}_{i,j} \in Y,\ k = 0, \ldots, N-1\}$.

Theorem 5.2. *For the Ising model on $\Sigma^{(3)}$, with periodic boundary conditions, we have: $R(\mathbf{c}^\pm; \mu_\pm) > 0$ if and only if $|M(\mu_\pm)| > 0$.*

Proof: If $R(\mathbf{c}^\pm; \mu_\pm) > 0$, then $|M(\mu_\pm)| > 0$ by Proposition 4.3 with $N = 3$. Conversely, if $|M(\mu_\pm)| > 0$, we can use Proposition 4.5 and prove that (23) holds. Indeed, for a given $Y \in \Gamma$, consider the set of all cylinders $\sigma_Y \subset D_Y^+$ and $\sigma_{\partial Y} \subset \partial C_Y^+$. Then, using Markov property, we have as in Theorem 5.1

$$\mu_{\Lambda_o}^+(C_Y^+) = \frac{1}{\mu_\Lambda(B^+)} \sum_{\sigma_{\partial Y} \subset \partial C_Y^+} \mu_\Lambda(B^+ \mid \sigma_{\partial Y})\mu_\Lambda(\sigma_{\partial Y}) \sum_{\sigma_Y \subset D_Y^+} \mu_\Lambda(\sigma_Y \mid \sigma_{\partial Y}),$$

and similarly for $\mu^{-}_{\Lambda_o}(C^{+}_{Y})$, hence

$$\frac{\mu^{+}_{\Lambda_o}(C^{+}_{Y})}{\mu^{-}_{\Lambda_o}(C^{+}_{Y})} \leq \sup_{\sigma_{\partial Y} \subset \partial C^{+}_{Y}} \frac{\mu_{\Lambda}(B^{+} \mid \sigma_{\partial Y})}{\mu_{\Lambda}(B^{-} \mid \sigma_{\partial Y})}. \tag{32}$$

Let us define

$$G_{\partial Y} := \{\tilde{\sigma} \in \Sigma : (\tilde{\sigma}_{i,j,0}, \tilde{\sigma}_{i,j,1}, \tilde{\sigma}_{i,j,2}) \in L\} \subset \partial C^{+}_{Y}, \tag{33}$$

where $L = \{(-1,-1,1), (-1,1,-1), (1,-1,-1)\}$. Relations (25) shows that the supremum in (32) is achieved on cylinders that are subset of $G_{\partial Y}$.

If R is the rotation operator defined as

$$(R\sigma)_{i,j,k} = \sigma_{i,j,k+1} \quad \forall (i,j,k) \in \mathbb{Z}^2 \times \{0,1,2\}, k = 0,1,2 \tag{34}$$

where $\sigma_{i,j,0} = \sigma_{i,j,3}$, then in a similar way used in Theorem 5.1, for $\sigma_{\partial Y} \subset G_{\partial Y}$ we obtain

$$\mu_{\Lambda}(B^{+} \mid \sigma_{\partial Y}) = \mu_{\Lambda}(B^{-} \mid (-R\sigma)_{\partial Y}) \leq \mu_{\Lambda}(B^{-} \mid \sigma_{\partial Y}). \tag{35}$$

The last inequality follows by the observation that if $\mathbf{c}_{i,j} \in \partial Y$, then $\sigma_{i,j,k} \leq (-R\sigma)_{i,j,k}$ holds for all $\sigma \in G_{\partial Y}$.

Thus

$$\limsup_{\Lambda \uparrow \mathbb{Z}^2 \times \{0,1,2\}} \sup_{\sigma_{\partial Y} \subset \partial C^{+}_{Y}} \frac{\mu_{\Lambda}(B^{+} \mid \sigma_{\partial Y})}{\mu_{\Lambda}(B^{-} \mid \sigma_{\partial Y})} \leq 1, \tag{36}$$

implying (23). This concludes the proof. □

Theorem 5.2 says that there exists a phase transition in the Ising model on $\Sigma^{(3)}$ with periodic boundary conditions if and only if there is a positive probability that the 3-vertex at the origin belongs to an infinite ($\mathbf{c}^{+}$)-cluster. Differently to the case of two slabs, we do not obtain an inequality between the 3-percolation probability and magnetization.

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